

# BOUNDARY REPRESENTATIONS OF HYPERBOLIC GROUPS

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**ABSTRACT.** Let  $\Gamma$  be a Gromov hyperbolic group, endowed with an arbitrary left-invariant hyperbolic metric, quasi-isometric to a word metric. The action of  $\Gamma$  on its boundary  $\partial\Gamma$  endowed with the Patterson-Sullivan measure  $\mu$ , after an appropriate normalization, gives rise to a unitary representation of  $\Gamma$  on  $L^2(\partial\Gamma, \mu)$ . We show that these representations are irreducible and weakly contained in the regular representation, and give criteria for their unitary equivalence in terms of the metrics on  $\Gamma$ .

## 1. INTRODUCTION

Any action of a group  $G$  on a measure space  $(X, \mu)$ , preserving the class of  $\mu$ , induces an action on the space of measurable functions on  $X$ . It can be normalized to obtain a unitary representation of  $G$  on  $L^2(X, \mu)$ . This construction generalizes the notion of a quasi-regular representation, which we obtain when  $X$  is a homogeneous space for  $G$ ; we will still call these representations quasi-regular.

Irreducibility of such quasi-regular representations is a mixing condition stronger than ergodicity—for non-ergodic actions the space  $L^2(X, \mu)$  decomposes into spaces of functions supported on the nontrivial invariant sets. There are many natural examples of such irreducible representations:

- the natural action of the group of diffeomorphisms of a manifold  $M$ , or some of its subgroups preserving additional structure on  $M$  [27, 20],
- the action of the Thompson's groups  $F$  and  $T$  on the unit interval and the unit circle [19],
- the action of a lattice of a Lie group on its Furstenberg boundary [4, 14],
- the action of the automorphism group of a regular tree on its boundary [15],
- the action of a free group on its boundary [16, 17],
- the action of the fundamental group of a compact strictly negatively curved Riemannian manifold  $M$  on the boundary of the universal cover of  $M$  [3].

The exact relation between irreducibility of the quasi-regular representation and the dynamical properties of the action of  $G$  on  $X$  is fully understood only in the case of discrete groups acting on discrete spaces [5, 11, 13]. The genuine quasi-regular representations are also better understood, via the notion of imprimitivity

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system [25]. For general locally compact groups, irreducibility of the quasi-regular representations was conjectured in [3] for another broad class of actions.

**Conjecture.** *For a locally compact group  $G$  and a spread-out probability measure  $\mu$  on  $G$ , the quasi-regular representation associated to the action of  $G$  on the  $\mu$ -boundary of  $G$  is irreducible.*

In this work we are studying the representations of hyperbolic groups associated with actions on their Gromov boundaries endowed with the Patterson-Sullivan measures. Following [3], we call them *boundary representations*. Our main result states that they are always irreducible. Moreover, when the metric on the group is quasi-isometrically perturbed, the class of the Patterson-Sullivan measure varies, thus leading to a potentially vast supply of non-equivalent irreducible representations. Indeed, we show that the only unitary equivalences between the boundary representations arise from rough similarities of the corresponding metrics. Our results thus generalize the work of Bader and Muchnik [3].

The irreducibility of the quasi-regular representations can also be viewed in a slightly different light. As far as we know, this is the first uniform construction of a family of irreducible unitary representations of an arbitrary hyperbolic group. In general, providing such constructions for large classes of groups, for which there is no structural description allowing to reduce the problem to some better understood cases, is a difficult task.

The line of our proof can be said to lie within bounded distance from the arguments of Bader and Muchnik, which we generalize to the setting of arbitrary hyperbolic groups, circumventing some of the difficulties they had to deal with. Basically, we construct a family of operators in the von Neumann algebra of the representation, analogous to the operators used in their approach. However, since they try to obtain them as weak operator limits of some arithmetic averages, in order to prove convergence they need to resort to a result of Margulis, describing the asymptotic behavior of the number of certain geodesic segments on a manifold. By using weighted averages and choosing suitable weights, we omit the necessity of knowing such asymptotics, and obtain a more self-contained, general, and simple proof.

Recently, Uri Bader has informed us about an unpublished work of Roman Muchnik, establishing irreducibility of quasi-regular representations of hyperbolic groups associated with their actions on Poisson boundaries of finitely supported symmetric random walks. This is also a special case of our result, which we explain in Section 8.2.

**1.1. Organization of the paper.** In Sections 2 and 3 we introduce some notational conventions and definitions from geometric group theory, discuss some basic results concerning hyperbolic groups and their boundaries, and finally define the class of representations we are going to deal with. All the geometry is contained in Section 4, where we explore some subsets of the group, estimate their growth and show that they are nicely distributed. Section 5 uses these estimates to construct certain operators in the von Neumann algebras of the boundary representations. In Section 6 we gather all the previous results into the proof of irreducibility of the boundary representations. We also explain why they are weakly contained in the regular representation. Section 7 contains the classification of the boundary representations with respect to unitary equivalence. Finally, in Section 8 we discuss two

examples with more explicitly defined groups and metrics. We have a closer look at the case of fundamental groups of negatively curved manifolds, and we also explain how the conjecture mentioned in the Introduction follows for a certain class of random walks on a hyperbolic group.

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## 2. PRELIMINARIES

In this section we introduce the basic notions associated with hyperbolic spaces and groups. We start by fixing some notational conventions for various kinds of estimates, which we will use throughout the paper in order to avoid the aggregation of non-essential constants and hopefully making the presentation more lucid. Then we introduce the basic terminology related to quasi-isometries, define hyperbolic spaces and groups, and finally, discuss the notion of the Gromov boundary. For details on these subjects see [10, Chapters III.H.1 and III.H.3].

**2.1. Estimates.** In the paper we will work with additive and multiplicative estimates. In order to avoid the escalation of constants coming from such estimates, we will suppress them using the following notation. Let  $f, g$  be functions on a set  $X$ . If there exists  $C > 0$  such that  $f(x) \leq Cg(x)$  for all  $x$ , we write  $f \prec g$ . If both  $f \prec g$  and  $g \prec f$  hold, we write  $f \asymp g$ . Analogously, for additive estimates,  $f \lesssim g$  if there exists  $c$  such that  $f \leq g + c$ , and  $f \approx g$  if both  $f \lesssim g$  and  $g \lesssim f$  hold. The variables in which the estimates are assumed to be uniform will be either clear from context or explicitly mentioned. Sometimes, to indicate that we do not care whether the estimate is uniform in some of the variables (which does not mean that we claim it is not), we write them as subscripts to the symbol of the corresponding estimate.

**2.2. Quasi-isometries.** Let  $(X, d_X)$  and  $(Y, d_Y)$  be metric spaces. A map  $\phi: X \rightarrow Y$  satisfying the condition

$$(2.1) \quad \frac{1}{L}d_X(p, q) - C \leq d_Y(\phi(p), \phi(q)) \leq Ld_X(p, q) + C$$

for all  $p, q \in X$  and some constants  $L \geq 1$  and  $C \geq 0$  is called a  $(L, C)$ -quasi-isometric embedding. If the image of  $\phi$  is a  $C$ -net in  $Y$ , i.e. its  $C$ -neighborhood covers  $Y$ , or equivalently, if there exists a quasi-isometric embedding  $\psi: Y \rightarrow X$ , called the *quasi-inverse* of  $\phi$ , such that  $d_X(x, \psi\phi(x))$  and  $d_Y(y, \phi\psi(y))$  are uniformly bounded functions on  $X$  and  $Y$  respectively, then  $\phi$  is a  $(L, C)$ -quasi-isometry. A  $(1, C)$ -quasi-isometry is called a *C-rough isometry*. A quasi-isometry  $\phi$  satisfying  $d_Y(\phi(x), \phi(y)) \approx Ld_X(x, y)$  with additive constant  $C$  is a  $(L, C)$ -rough similarity.

A  $(L, C)$ -quasi-geodesic in  $X$  is a  $(L, C)$ -quasi-isometric embedding  $\gamma: \mathbb{R} \rightarrow X$ . Similarly one defines quasi-geodesic rays and segments, and their roughly geodesic variants. We say that  $X$  is a  $(L, C)$ -quasi-geodesic space, if any two points in  $X$  can be joined by a  $(L, C)$ -quasi-geodesic segment. A *C-roughly geodesic space* is defined in the same manner. We will later fix the constants  $L$  and  $C$  and suppress them from notation.

**2.3. Hyperbolic spaces and groups.** Let  $(X, d)$  be a metric space. For any basepoint  $o \in X$  one defines the Gromov product  $(\cdot, \cdot)_o: X \times X \rightarrow [0, \infty)$  with respect to  $o$  as

$$(2.2) \quad (x, y)_o = \frac{1}{2}(d(x, o) + d(y, o) - d(x, y)).$$

A different choice of the basepoint leads to another Gromov product, satisfying

$$(2.3) \quad |(x, y)_o - (x, y)_p| \leq d(o, p).$$

If the Gromov product on  $X$  satisfies the estimate

$$(2.4) \quad (x, y)_o \geq \min\{(x, z)_o, (y, z)_o\}$$

for some (equivalently, for every—but with a different constant) basepoint  $o \in X$ , the space  $X$  is said to be *hyperbolic*. We may iterate (2.4), to obtain

$$(2.5) \quad (x_1, x_n)_p \geq \min\{(x_1, x_2)_p, (x_2, x_3)_p, \dots, (x_{n-1}, x_n)_p\},$$

with constants depending only on  $n$ . The property of being hyperbolic is preserved by quasi-isometries within the class of geodesic spaces. In case of general metric spaces, it is possible to quasi-isometrically perturb a hyperbolic metric and obtain a non-hyperbolic one (see [8, Proposition A.11]).

A finitely generated group  $\Gamma$  is hyperbolic if its Cayley graph with respect to some finite set of generators is hyperbolic. As Cayley graphs of a given group are geodesic and quasi-isometric to each other, this notion does not depend on the generating set. The quasi-isometric metrics induced on the group by the path metrics on its Cayley graphs are called the *word metrics*. We will denote by  $\mathcal{D}(\Gamma)$  the class of all hyperbolic left-invariant metrics on  $\Gamma$  (not necessarily coming from an action on a geodesic space), quasi-isometric to a word metric (when we say that two metrics on the same space are quasi-isometric, we mean that the identity map is a quasi-isometry). Finally, a hyperbolic group is *non-elementary* if it does not contain a cyclic subgroup of finite index.

**2.4. The Gromov boundary.** Now, assume that  $X$  is hyperbolic and has a fixed basepoint  $o \in X$ , which we will omit in the notation for the Gromov product. We will also denote  $|x| = d(x, o)$ . A sequence  $(x_n) \subset X$  *tends to  $\infty$*  if

$$(2.6) \quad \lim_{i, j \rightarrow \infty} (x_i, x_j) = \infty.$$

Two such sequences  $(x_n)$  and  $(y_n)$  are *equivalent* if  $\lim_{n \rightarrow \infty} (x_n, y_n) = \infty$ . By (2.3) these notions are independent of the basepoint. The boundary of  $X$ , denoted  $\partial X$ , is the set of equivalence classes of sequences tending to infinity. The space  $\overline{X} = X \cup \partial X$  can be given a natural topology making it a compactification of  $X$ , on which the isometry group  $\text{Isom}(X)$  acts by homeomorphisms.

The Gromov product can be extended (in a not necessarily continuous way) to  $\overline{X}$  in such a way that the estimate (2.4) is still satisfied (with different constants). One simply represents elements of  $X$  as constant sequences, and for  $x, y \in \overline{X}$  defines

$$(2.7) \quad (x, y) = \sup \liminf_{i, j \rightarrow \infty} (x_i, y_j),$$

where the supremum is taken over all representatives  $(x_i)$  and  $(y_i)$  of  $x$  and  $y$ . A sequence  $(x_i) \subset X$  converges to  $\xi \in \partial X$  if and only if  $(x_i, \xi) \rightarrow \infty$ , so in particular,

representatives of  $\zeta$  are exactly the sequences in  $X$  converging to  $\zeta$ . By [10, Remark 3.17], we have

$$(2.8) \quad \liminf_{i,j \rightarrow \infty} (x_i, y_j) \approx (x, y)$$

whenever  $x_i \rightarrow x$  and  $y_i \rightarrow y$ .

The topology of  $\partial X$  is metrizable. For sufficiently small  $\epsilon > 0$  there exists a metric  $d_\epsilon$  on  $\partial X$ , compatible with its topology, satisfying

$$(2.9) \quad d_\epsilon(\zeta, \eta) \asymp_\epsilon e^{-\epsilon(\zeta, \eta)}.$$

Such a metric is called a *visual metric*.

We will later use the fact that for a hyperbolic group  $\Gamma$  the only  $\Gamma$ -equivariant homeomorphism  $\phi$  of  $\partial\Gamma$  is the identity map. It follows from the fact that any element of  $\Gamma$  of infinite order has exactly one attracting point in  $\partial\Gamma$ , which is therefore fixed by  $\phi$ , and the attracting points of all such elements form a dense subset [23, Proposition 4.2 and Theorem 4.3].

### 3. THE GEOMETRIC SETTING

In this section we describe some ways to deal with non-geodesic hyperbolic metrics. As the representations we will consider depend on the metric on the group, this will allow to investigate a class of representations much wider than those obtained from the word metrics. Everything we need in this regard is contained in the papers [8, 9].

In [8] the notions of a quasi-ruler and quasi-ruled space are introduced, and the fundamental properties of the Patterson-Sullivan measures for quasi-ruled hyperbolic spaces, generalizing the results of [12], which apply only to metrics coming from proper actions on geodesic spaces, are developed. The article [9] studies boundaries of almost geodesic hyperbolic spaces, and is a useful reference for some basic lemmas.

It turns out that the classes of hyperbolic quasi-ruled spaces and hyperbolic almost geodesic spaces are the same and equal to the class of roughly geodesic hyperbolic spaces. We discuss the notion of a quasi-ruled space only in order to formulate Theorem 3.1. Afterwards, all the arguments will be based on the notion of a rough geodesic.

**3.1. Roughly geodesic hyperbolic spaces.** For  $\tau \geq 0$  a  $\tau$ -*quasi-ruler* is a quasi-geodesic  $\gamma: \mathbb{R} \rightarrow X$  satisfying for all  $s < t < u$  the condition

$$(3.1) \quad (\gamma(s), \gamma(u))_{\gamma(t)} \leq \tau.$$

The space  $X$  is said to be  $(L, C, \tau)$ -*quasi-ruled* if it is a  $(L, C)$ -quasi-geodesic space, and every  $(L, C)$ -quasi-geodesic is a  $\tau$ -quasi-ruler. By [8, Theorem A.1], if  $\phi: X \rightarrow Y$  is a quasi-isometry with  $X$  hyperbolic and geodesic, then  $Y$  is hyperbolic if and only if it is quasi-ruled. It follows that for a hyperbolic group  $\Gamma$ , all the metrics in the class  $\mathcal{D}(\Gamma)$  are quasi-ruled.

By [8, Lemma A.2], for every  $L, C$ , and  $\tau$  there exists  $K > 0$  such that every  $(L, C, \tau)$ -quasi-ruled space is  $K$ -roughly geodesic. On the other hand, it is clear that a  $K$ -roughly geodesic space is  $(1, K, 3K/2)$ -quasi-ruled. By [9, Proposition 5.2(1)], the hyperbolic spaces studied therein are also exactly the roughly geodesic hyperbolic spaces.

Now suppose that  $X$  is a roughly geodesic hyperbolic space. Every roughly geodesic ray  $\gamma$  in  $X$  converges to an endpoint  $\gamma(\infty)$  in the boundary. The converse statement is also true, i.e. every point in  $\partial X$  is the endpoint of some  $K$ -roughly geodesic ray, where  $K$  depends only on  $X$  [9, Proposition 5.2(2)]. From now on, when we use the terms *roughly geodesic segment/ray* without specifying the constant, we always think of the universal constants from the definition of a roughly geodesic space and the remark above.

By [9, Proposition 5.5], if  $\phi: X \rightarrow Y$  is a  $(L, C)$ -quasi-isometry of roughly geodesic hyperbolic spaces, their Gromov products satisfy estimates similar to the metrics, i.e.

$$(3.2) \quad \frac{1}{L}(x, y)_z \lesssim (\phi(x), \phi(y))_{\phi(z)} \lesssim L(x, y)_z$$

holds uniformly for all  $x, y, z \in X$ . As a consequence, for a hyperbolic group  $\Gamma$  all the metrics in  $\mathcal{D}(\Gamma)$  give rise to exactly the same boundary.

**3.2. Quasi-conformal measures.** Consider a roughly geodesic hyperbolic space  $X$  with a basepoint  $o \in X$ , and a non-elementary hyperbolic group  $\Gamma \subseteq \text{Isom}(X)$  which acts on  $X$  properly and cocompactly. A measure  $\mu$  on  $(\partial X, d_\epsilon)$  is said to be  $\Gamma$ -quasi-conformal of dimension  $D$  if it is quasi-invariant under the action of  $\Gamma$ , and the corresponding Radon-Nikodym derivatives satisfy the estimate

$$(3.3) \quad \frac{dg_*\mu}{d\mu}(\xi) \asymp e^{\epsilon D(2(go, \xi) - d(o, go))}$$

uniformly in  $\xi$  and  $g$ . Since  $d(o, go) \approx_{o,p} d(p, gp)$ , this notion is independent of the choice of  $o$ . Moreover, being  $\Gamma$ -quasi-conformal does not depend on  $\epsilon$ , as for different values of  $\epsilon$  only the dimension  $D$  changes. Finally,  $\mu$  is *Ahlfors regular of dimension  $D$*  if it satisfies the estimate

$$(3.4) \quad \mu(B_{\partial X}(\xi, r)) \asymp r^D$$

uniformly in  $\xi$  and  $r \leq \text{diam } \partial X$ . In particular, since  $\partial X$  is compact, any Ahlfors regular measure on  $\partial X$  is finite.

Recall that the Hausdorff measure of a metric space  $Y$  is defined as follows. First, for  $\alpha \geq 0$  one defines the  $\alpha$ -dimensional Hausdorff measure  $\mathcal{H}_\alpha$  as

$$(3.5) \quad \mathcal{H}_\alpha(E) = \lim_{\theta \rightarrow 0^+} \inf \left\{ \sum_i (\text{diam } U_i)^\alpha : E \subseteq \bigcup_i U_i \text{ and } \text{diam } U_i \leq \theta \right\}$$

for every Borel set  $E \subseteq Y$ . Then, the Hausdorff dimension of  $Y$  is the number

$$(3.6) \quad \dim_H Y = \inf \{ \alpha : \mathcal{H}_\alpha(Y) = 0 \} = \sup \{ \alpha : \mathcal{H}_\alpha(Y) = \infty \}.$$

The Hausdorff measure on  $Y$  is the  $(\dim_H Y)$ -dimensional Hausdorff measure.

Now, take  $x \in X$  and denote

$$(3.7) \quad D = \limsup_{R \rightarrow \infty} \frac{1}{\epsilon R} \log |B_X(x, R) \cap \Gamma x|,$$

and  $\omega = e^{D\epsilon}$ . We then have the following.

**Theorem 3.1** ([8, Theorem 2.3]). *Suppose that  $X$  is a proper roughly geodesic hyperbolic space. Then the Hausdorff dimension of  $(\partial X, d_\epsilon)$  is equal to  $D$ , and the corresponding Hausdorff measure  $\mu$  is  $\Gamma$ -quasi-conformal of dimension  $D$  and Ahlfors regular of dimension  $D$ . Furthermore, any  $\Gamma$ -quasi-conformal measure  $\mu'$  on  $\partial X$  is equivalent to  $\mu$  with Radon-Nikodym derivative  $d\mu'/d\mu \asymp 1$  a.e., and  $|B_X(x, R) \cap \Gamma x| \asymp \omega^R$ .*

In particular, this theorem implies that the quasi-conformal measures associated to different choices of  $\epsilon$  are equivalent, so the above considerations lead to a unique measure class on  $\partial X$  (in fact a class of the finer relation of equivalence with Radon-Nikodym derivatives bounded away from 0 and  $\infty$ ), depending only on the metric  $d$ , called the *Patterson-Sullivan class*. Also, by Ahlfors regularity, the boundary has no isolated points.

We say that a measure class preserving action of a group  $G$  on a measure space  $(X, \nu)$  is *doubly ergodic*, if the induced diagonal action of  $G$  on  $(X^2, \mu^2)$  is ergodic. In the classification of the boundary representations, double ergodicity of Patterson-Sullivan measures will be crucial. This result is known to experts, but apparently the proof has never been written down. It was communicated to us by Uri Bader that the full proof will appear in the forthcoming paper [2], joint with Alex Furman.

**3.3. Boundary representations.** We now fix some notation for the rest of the paper. Let  $\Gamma$  be a non-elementary hyperbolic group. Fix a metric  $d \in \mathcal{D}(\Gamma)$ , and choose  $1 \in \Gamma$  as the basepoint. Since  $\Gamma$  acts on itself by isometries freely and co-compactly, we are in the setting of Section 3.2. Pick a sufficiently small  $\epsilon > 0$ , and let  $D$  be the Hausdorff dimension of  $(\partial\Gamma, d_\epsilon)$ . Denote by  $\mu$  the corresponding Hausdorff measure. We may normalize  $\mu$  and  $d_\epsilon$  in such a way that  $\mu(\partial\Gamma) = 1$  and  $\text{diam } \partial\Gamma = 1$ . By changing  $\epsilon$ , we may also assume that  $D > 1$ . Now, denote

$$(3.8) \quad P_g(\xi) = \frac{dg_*\mu}{d\mu}(\xi) \asymp \omega^{2(g, \xi) - |g|}.$$

The *boundary representation*  $\pi$  of  $\Gamma$  with respect to  $\mu$  is the unitary representation of  $\Gamma$  on the Hilbert space  $L^2(\partial\Gamma, \mu)$  given by

$$(3.9) \quad [\pi(g)\phi](\xi) = P_g^{1/2}(\xi)\phi(g^{-1}\xi)$$

for  $\phi \in L^2(\partial\Gamma, \mu)$  and  $g \in \Gamma$ . If we take a measure  $\nu$  equivalent to  $\mu$ , then the unitary isomorphism  $T_{\mu\nu}: L^2(\partial\Gamma, \mu) \rightarrow L^2(\partial\Gamma, \nu)$  defined by

$$(3.10) \quad T_{\mu\nu}\phi = \left(\frac{d\mu}{d\nu}\right)^{1/2} \phi$$

intertwines the corresponding boundary representations. We therefore obtain a unique representation of  $\Gamma$  associated to the class of  $\Gamma$ -quasi-conformal measures on  $\partial\Gamma$  with respect to  $d$ .

#### 4. SHADOWS AND CONES

In this section we will work in  $\Gamma$  in order to estimate the cardinalities of some of its subsets. First, we introduce the classical notion of the shadow cast by an element of the group onto its boundary. Then, for a ball  $B$  in the boundary, we define the cone over  $B$  as the set of all elements  $\Gamma$  whose shadows intersect  $B$ . It turns out that the growth of such a cone behaves as one could expect, i.e. its intersection with a large ball in  $\Gamma$  comprises approximately  $\mu(B)$  of the ball's volume.

We then move on to define double shadows in  $\partial\Gamma^2$ . A double shadow of  $g$  is the product of the shadows of  $g$  and  $g^{-1}$ . We show that such double shadows form a nice cover of  $\partial\Gamma^2$ , just as in the case of normal shadows and  $\partial\Gamma$ .

**4.1. Shadows.** We begin by observing that for any element of  $\Gamma$  there exists a geodesic ray emanating from 1 and passing within a uniform distance. In terms of the Gromov product this can be stated as follows.

**Lemma 4.1.** *The estimate*

$$(4.1) \quad \sup_{\xi \in \partial\Gamma} (g, \xi) \approx |g|$$

holds uniformly for  $g \in \Gamma$ .

*Proof.* We get the upper estimate  $\sup(g, \xi) \leq |g|$  from the triangle inequality.

The Gromov product on  $\Gamma$  satisfies the identity  $(g, h) + (g^{-1}, g^{-1}h) = |g|$ , which, after extension to  $\bar{\Gamma}$ , takes the form

$$(4.2) \quad (g^{-1}, g^{-1}\xi) \approx |g| - (g, \xi).$$

If we fix two distinct points  $\xi_1, \xi_2 \in \partial\Gamma$ , then

$$(4.3) \quad \max_i (g^{-1}, g^{-1}\xi_i) \approx |g| - \min_i (g, \xi_i) \gtrsim |g| - (\xi_1, \xi_2),$$

which gives the estimate from below for  $g^{-1}$ .  $\square$

Using Lemma 4.1, for every  $g \in \Gamma$  we may fix  $\hat{g} \in \partial\Gamma$  such that  $(g, \hat{g}) \approx |g|$ . We will also denote by  $\check{g}$  the point corresponding to  $g^{-1}$ . The point  $\hat{g}$  plays the same role as the endpoint of the geodesic ray starting at the basepoint and passing through  $g$  in a CAT(0) space. In particular, we have

$$(4.4) \quad (\xi, g) \approx \min\{|g|, (\hat{g}, \xi)\}$$

for all  $\xi \in \Gamma$ . This estimate will be usually used in the form

$$(4.5) \quad \omega^{(\xi, g)} \asymp \min\{\omega^{|g|}, d_\epsilon(\hat{g}, \xi)^{-D}\}.$$

By Theorem 3.1, the growth of  $\Gamma$  satisfies  $|B_\Gamma(1, R)| \asymp \omega^R$ . Fix  $r > 0$  and for  $R > 0$  define the annulus

$$(4.6) \quad A_R = \{g \in \Gamma : R - r \leq |g| \leq R + r\},$$

We will assume that  $r$  is sufficiently large for the following three conditions to hold:

- (1) the annuli  $A_R$  satisfy the growth estimate  $|A_R| \asymp \omega^R$ ,
- (2) for any roughly geodesic ray  $\gamma$  from 1 to  $\xi \in \partial\Gamma$  we have  $\gamma(R) \in A_R$ ,
- (3)  $r$  satisfies the bound obtained in the proof of Proposition 4.5, ensuring that the elements of  $\Gamma$  of length approximately  $R$  constructed therein are in  $A_R$ .

For  $\rho > 0$  define the *shadow*  $\Sigma(g, \rho)$  of  $g$  as the closed ball

$$(4.7) \quad \Sigma(g, \rho) = B_{\partial\Gamma}(\hat{g}, e^{-\epsilon(|g| - \rho)}).$$

The following fundamental property of shadows is classical, and we include its very short proof. This standard lemma has also a second part, saying that the multiplicity of the cover of  $\partial\Gamma$  by shadows is uniformly bounded in  $R$ , but we will not need this statement.

**Lemma 4.2.** *For sufficiently large  $\rho$ , the family of shadows  $\{\Sigma(g, \rho) : g \in A_R\}$  is a cover of  $\partial\Gamma$  for any  $R \geq 0$ .*



*Proof.* For  $\xi \in \partial\Gamma$  take a roughly geodesic ray  $\gamma$  from 1 to  $\xi$ . Then  $g = \gamma(R) \in A_R$  and we have  $(\hat{g}, \xi) \gtrsim \min\{(\hat{g}, g), (g, \xi)\} \approx R$ , so for sufficiently large  $\rho$  we get  $\xi \in \Sigma(g, \rho)$ .  $\square$

We may now define  $\Sigma(g) = \Sigma(g, \rho)$  with  $\rho$  sufficiently large to satisfy the conclusion of Lemma 4.2.

**4.2. Cones over balls in the boundary.** For  $\xi \in \partial\Gamma$  and  $\rho > 0$  define the *cone over*  $B_{\partial\Gamma}(\xi, e^{-\epsilon\rho})$  as

$$(4.8) \quad C(\xi, \rho) = \{g \in \Gamma : \Sigma(g) \cap B_{\partial\Gamma}(\xi, e^{-\epsilon\rho}) \neq \emptyset\},$$

and denote

$$(4.9) \quad C_R(\xi, \rho) = A_R \cap C(\xi, \rho).$$

**Lemma 4.3.** *The growth of the cone  $C(\xi, \rho)$  satisfies the estimates*

$$(4.10) \quad \omega^{R-\rho} \prec |C_R(\xi, \rho)| \prec \omega^R$$

*uniformly in  $R, \rho, \xi$ . When  $R \geq \rho$ , the better estimate*

$$(4.11) \quad |C_R(\xi, \rho)| \asymp \omega^{R-\rho}$$

*holds.*

*Proof.* The upper bound  $|C_R(\xi, \rho)| \prec \omega^R$  follows from the estimate on  $|A_R|$ . For the lower bound, observe that, by Lemma 4.2, the shadows  $\Sigma(g)$  of  $g \in C_R(\xi, \rho)$  cover the ball  $B(\xi, e^{-\epsilon\rho})$ . Hence,

$$(4.12) \quad \omega^{-\rho} \asymp \mu(B_{\partial\Gamma}(\xi, e^{-\epsilon\rho})) \leq \sum_{g \in C_R(\xi, \rho)} \mu(\Sigma(g)) \asymp |C_R(\xi, \rho)| \omega^{-R},$$

so  $|C_R(\xi, \rho)| \succ \omega^{R-\rho}$ .

Now, assume that  $R \geq \rho$ . Let  $\gamma$  be a roughly geodesic ray from 1 to  $\xi$ . If  $g \in C_R(\xi, \rho)$ , we may pick some  $\eta \in \Sigma(g) \cap B_{\partial\Gamma}(\xi, e^{-\epsilon\rho})$ . We then have

$$(4.13) \quad (g, \gamma(\rho)) \gtrsim \min\{(g, \hat{g}), (\hat{g}, \eta), (\eta, \xi), (\xi, \gamma(\rho))\} \gtrsim \min\{R, R, \rho, \rho\} = \rho,$$

and in consequence

$$(4.14) \quad d(g, \gamma(\rho)) \lesssim R - \rho.$$

Hence,  $C_R(\xi, \rho) \subseteq B_X(\gamma(\rho), R - \rho + C)$  for some constant  $C$ , and the last estimate follows from the bound on the growth of  $\Gamma$ .  $\square$

**4.3. Shadows in the square of the boundary.** The next lemma will allow us to understand the distribution of the points  $(\hat{g}, \hat{g})$  in  $\partial\Gamma^2$ . It generalizes the observation that if we take two elements  $g, h$  of a nonabelian free group, expressed in the standard generators, then after possibly changing the last letter of  $g$ , there is no cancellation in the product  $gh$ . We thought that such a natural result should be well-known, but to our surprise we did not find it in any of our standard references for hyperbolic groups. Therefore we present it together with its full proof.

**Lemma 4.4.** *Let  $\Gamma$  be a non-elementary hyperbolic group, endowed with a metric  $d \in \mathcal{D}(\Gamma)$ . There exists  $\tau > 0$  such that for any  $g_0, h \in \Gamma$  one can find  $g \in B_\Gamma(g_0, \tau)$  such that  $|gh| \geq |g| + |h| - 2\tau$ .*

*Proof.* For every  $g \in \Gamma$  fix a roughly geodesic segment  $\gamma_g: [0, |g|] \rightarrow \Gamma$  joining 1 to  $g$ , and its reverse  $\bar{\gamma}_g(t) = \gamma_g(|g| - t)$ . Now, take any  $\tau > 0$  and suppose that for all  $g \in B_\Gamma(g_0, \tau)$  we have the opposite inequality  $|gh| < |g| + |h| - 2\tau$ , or equivalently,  $(1, gh)_g > \tau$ . In particular, this implies that  $|g|, |h| > \tau$ , and we may compute (with estimates being uniform in  $\tau$ )

$$(4.15) \quad (\bar{\gamma}_g(\tau), g\gamma_h(\tau))_g \geq \min\{(\bar{\gamma}_g(\tau), 1)_g, (1, gh)_g, (gh, g\gamma_h(\tau))_g\} \geq \tau,$$

and in consequence,  $d(\bar{\gamma}_g(\tau), g\gamma_h(\tau)) \approx 0$ . On the other hand,  $(g, g_0) \geq |g| - \tau$ , so

$$(4.16) \quad (\bar{\gamma}_g(\tau), \gamma_{g_0}(|g| - \tau)) \geq \min\{(\bar{\gamma}_g(\tau), g), (g, g_0), (g_0, \gamma_{g_0}(|g| - \tau))\} \approx |g| - \tau,$$

and thus  $d(\bar{\gamma}_g(\tau), \gamma_{g_0}(|g| - \tau)) \approx 0$ . Finally, we obtain

$$(4.17) \quad d(g\gamma_h(\tau), \gamma_{g_0}(|g| - \tau)) \leq d(g\gamma_h(\tau), \bar{\gamma}_g(\tau)) + d(\bar{\gamma}_g(\tau), \gamma_{g_0}(|g| - \tau)) \approx 0.$$

It follows that the injective map  $g \mapsto g\gamma_h(\tau)$  sends the ball  $B_\Gamma(g_0, \tau)$  into a fixed radius neighborhood of the interval  $\gamma_{g_0}([|g_0| - 2\tau, |g_0|])$ . Since  $\Gamma$  is non-elementary, the volume of the ball grows exponentially with  $\tau$ , while the neighborhood of the interval  $\gamma_{g_0}([|g_0| - 2\tau, |g_0|])$  has linear growth, hence for sufficiently large  $\tau$ , independent of  $g_0$  and  $h$ , we obtain a contradiction.  $\square$

For  $\rho > 0$  we now define the *double shadow* of  $g \in \Gamma$  as

$$(4.18) \quad \Sigma_2(g, \rho) = B_{\partial\Gamma}(\hat{g}, e^{-\epsilon(|g|/2 - \rho)}) \times B_{\partial\Gamma}(\check{g}, e^{-\epsilon(|g|/2 - \rho)}) \subseteq \partial\Gamma^2.$$

Thanks to the factor of  $1/2$  in the exponent, the measure of a double shadow of  $g \in A_R$  is approximately proportional to  $1/|A_R|$ . Just as ordinary shadows, the double shadows of elements of  $A_R$  form a cover.

**Proposition 4.5.** *For sufficiently large  $\rho > 0$  the family  $\{\Sigma_2(g, \rho) : g \in A_R\}$  of double shadows is a cover of  $\partial\Gamma^2$  for all  $R > 0$ .*

*Proof.* Take  $(\xi_1, \xi_2) \in \partial\Gamma^2$ , and let  $\gamma_i$  be a roughly geodesic ray from 1 to  $\xi_i$ . Put  $g_i = \gamma_i(R/2)$ . By Lemma 4.4 there exist a universal constant  $\tau$  and  $g \in B_\Gamma(g_1, \tau)$  such that  $|gg_2^{-1}| \approx R$ , and, as it was mentioned in the definition of the annulus  $A_R$ , we may assume that its thickness is sufficiently large for it to contain the element  $gg_2^{-1}$ . We have  $(gg_2^{-1}, g) \approx R/2$  and  $(g_2g^{-1}, g_2) \approx R/2$ , and in consequence

$$(4.19) \quad (gg_2^{-1}, \xi_1) \geq \min\{(gg_2^{-1}, g), (g, g_1), (g_1, \xi_1)\} \geq R/2$$

and

$$(4.20) \quad (g_2g^{-1}, \xi_2) \geq \min\{(g_2g^{-1}, g_2), (g_2, \xi_2)\} \geq R/2.$$

Hence, for sufficiently large  $\rho$ , the double shadow  $\Sigma_2(gg_2^{-1}, \rho)$  contains the pair  $(\xi_1, \xi_2)$ .  $\square$

Similarly as in the case of shadows, we will denote  $\Sigma_2(g) = \Sigma_2(g, \rho)$  for some fixed  $\rho$  sufficiently large for Proposition 4.5 to hold.

## 5. OPERATORS IN THE POSITIVE CONE

By the *positive cone of the representation*  $\pi$  we will understand the weak operator closure of the cone spanned by  $\pi(\Gamma)$  in  $\mathcal{B}(L^2(\partial\Gamma, \mu))$ . The purpose of this section is to prove Proposition 5.4, which states that operators arising from positive kernels in  $L^\infty(\partial\Gamma^2)$  are contained in the positive cone of  $\pi$ . The operators in question will be constructed as weak operator limits of sequences of weighted averages of normalized operators  $\pi(g)$  with  $g \in A_R$ . Convergence will be first tested on Lipschitz functions, and then established using density of Lipschitz functions in  $L^2$  and uniform boundedness of the averages.

**5.1. Uniform boundedness of averages of  $P_g^{1/2}$ .** Let us define

$$(5.1) \quad \tilde{P}_g = P_g^{1/2} / \|P_g^{1/2}\|_1.$$

We begin by finding an estimate for the norm  $\|P_g^{1/2}\|_1$ , and an approximation of the function  $\tilde{P}_g$ .

**Lemma 5.1.** *The  $L^1$ -norms of  $P_g^{1/2}$  satisfy the estimate*

$$(5.2) \quad \|P_g^{1/2}\|_1 \asymp \omega^{-|g|/2}(1 + |g|)$$

*uniformly in  $g$ , and in consequence*

$$(5.3) \quad \tilde{P}_g(\xi) \asymp \frac{\omega(g, \xi)}{1 + |g|}.$$

*Proof.* By estimate (4.4), we have

$$(5.4) \quad P_g^{1/2}(\xi) \asymp \omega^{(g, \xi) - |g|/2} \asymp \omega^{-|g|/2} \min\{\omega^{|g|}, d_\epsilon(\hat{g}, \xi)^{-D}\}.$$

Using Ahlfors regularity, we calculate

$$(5.5) \quad \begin{aligned} \omega^{|g|/2} \|P_g^{1/2}\|_1 &\asymp \int_{\partial\Gamma} \min\{\omega^{|g|}, d_\epsilon(\hat{g}, \xi)^{-D}\} d\mu(\xi) = \\ &= 1 + \int_1^{\omega^{|g|}} \mu\{\xi : d_\epsilon(\hat{g}, \xi)^{-D} > t\} dt \asymp 1 + \int_1^{\omega^{|g|}} t^{-1} dt \asymp 1 + |g|. \quad \square \end{aligned}$$

Now we prove the crucial result, stating that the averages of the functions  $\tilde{P}_g$  over  $A_R$  are uniformly bounded in the  $L^\infty$  norm. Later on, the problem of uniform boundedness of weighted averages of suitably normalized operators  $\pi(g)$  will be reduced to this estimate.

**Proposition 5.2.** *The estimate*

$$(5.6) \quad \sum_{g \in A_R} \tilde{P}_g(\eta) \prec \omega^R$$

*holds uniformly in  $R$  and  $\eta$ .*

*Proof.* By Lemma 5.1 and (4.5) we have

$$(5.7) \quad \sum_{g \in A_R} \tilde{P}_g(\eta) \asymp \frac{1}{(1 + R)} \sum_{g \in A_R} \min\{\omega^R, d_\epsilon(\hat{g}, \eta)^{-D}\}.$$

The sum on the right can be estimated as

$$\begin{aligned}
 (5.8) \quad \sum_{g \in A_R} \min\{\omega^R, d_\epsilon(\hat{g}, \eta)^{-D}\} &\prec \omega^R + \int_1^{\omega^R} \left| \{g \in A_R : d_\epsilon(\hat{g}, \eta)^{-D} > t\} \right| dt = \\
 &= \omega^R + \int_1^{\omega^R} |C_R(\eta, \log t / D\epsilon)| dt
 \end{aligned}$$

But for  $1 < t < \omega^R$  we have  $0 < \log t / D\epsilon < R$ , so we may use Lemma 4.3 to obtain

$$(5.9) \quad \int_1^{\omega^R} |C_R(\eta, \log t / D\epsilon)| dt \asymp \int_1^{\omega^R} \omega^R t^{-1} dt = \omega^R R \log \omega,$$

which ends the proof.  $\square$

**5.2. Approximation on the space of Lipschitz functions.** Denote by  $\text{Lip}(\partial\Gamma)$  the vector space of Lipschitz functions on  $(\partial\Gamma, d_\epsilon)$ . Let  $\lambda(\phi)$  be the Lipschitz constant of  $\phi \in \text{Lip}(\partial\Gamma)$ . By the Lebesgue differentiation theorem [21, Theorem 1.8], valid in particular for any Ahlfors regular metric measure space, the characteristic functions of balls span a dense subspace of  $L^2(\partial\Gamma, \mu)$ , and since they can be approximated by Lipschitz functions, it follows that  $\text{Lip}(\partial\Gamma)$  is a dense subspace of  $L^2(\partial\Gamma, \mu)$ .

Define the normalized operator  $\tilde{\pi}(g) = \pi(g) / \|P_g^{1/2}\|_1$ . Since

$$(5.10) \quad \|P_g^{1/2}\|_1 = \langle \pi(g)\mathbf{1}, \mathbf{1} \rangle = \langle \mathbf{1}, \pi(g^{-1})\mathbf{1} \rangle = \|P_{g^{-1}}^{1/2}\|_1,$$

the operators  $\tilde{\pi}(g)$  satisfy  $\tilde{\pi}(g)^* = \tilde{\pi}(g^{-1})$ . Moreover, as the next lemma shows, it turns out that on Lipschitz functions  $\tilde{\pi}(g)$  can be approximated in a particularly nice way.

**Lemma 5.3.** *For  $\phi, \psi \in \text{Lip}(\partial\Gamma)$  we have*

$$(5.11) \quad \left| \langle \tilde{\pi}(g)\phi, \psi \rangle - \phi(\check{g}) \overline{\psi(\hat{g})} \right| \prec \frac{\lambda(\phi) \|\psi\|_\infty + \lambda(\psi) \|\phi\|_\infty}{(1 + |g|)^{1/D}},$$

uniformly in  $g, \phi$ , and  $\psi$ .

*Proof.* We have

$$\begin{aligned}
 (5.12) \quad \left| \langle \tilde{\pi}(g)\phi, \psi \rangle - \phi(\check{g}) \overline{\psi(\hat{g})} \right| &\leq \\
 &\leq |\langle \tilde{\pi}(g)\phi, \psi - \psi(\hat{g})\mathbf{1} \rangle| + \left| \overline{\psi(\hat{g})} (\langle \phi, \tilde{\pi}(g^{-1})\mathbf{1} \rangle - \phi(\check{g})) \right| \leq \\
 &\leq \|\phi\|_\infty \int_{\partial\Gamma} \tilde{P}_g(\xi) |\psi(\xi) - \psi(\hat{g})| d\mu(\xi) + \\
 &\quad + \|\psi\|_\infty \int_{\partial\Gamma} \tilde{P}_{g^{-1}}(\xi) |\phi(\xi) - \phi(\check{g})| d\mu(\xi).
 \end{aligned}$$

Both terms are similar, so we will estimate only the first one. Since  $\psi$  is Lipschitz, we get

$$(5.13) \quad \int_{\partial\Gamma} \tilde{P}_g(\xi) |\psi(\xi) - \psi(\hat{g})| d\mu(\xi) \leq \lambda(\psi) \int_{\partial\Gamma} \tilde{P}_g(\xi) d_\epsilon(\hat{g}, \xi) d\mu(\xi),$$

and by integrating separately on some ball  $B = B_{\partial\Gamma}(\hat{g}, \theta)$  and its complement, we obtain

$$(5.14) \quad \int_B \tilde{P}_g d_\epsilon(\hat{g}, \xi) d\mu(\xi) \leq \theta \|\tilde{P}_g\|_1 = \theta$$

and

$$(5.15) \quad \int_{\partial\Gamma \setminus B} \tilde{P}_g d_\epsilon(\hat{g}, \xi) d\mu(\xi) \prec \int_{\partial\Gamma \setminus B} \frac{d_\epsilon(\hat{g}, \xi)^{1-D}}{1 + |g|} d\mu(\xi) \prec \frac{\theta^{1-D}}{1 + |g|},$$

since  $D > 1$ . We finish by taking  $\theta = (1 + |g|)^{-1/D}$ .  $\square$

**5.3. Constructing operators in the positive cone of the representation.** For a function  $K \in L^\infty(\partial\Gamma^2, \mu^2)$ , define the operator  $T_K$  with kernel  $K$  by

$$(5.16) \quad \langle T_K \phi, \psi \rangle = \int_{\partial\Gamma^2} \phi(\xi) \overline{\psi(\eta)} K(\xi, \eta) d\mu^2(\xi, \eta).$$

**Proposition 5.4.** *The operator  $T_K$  with kernel  $K \geq 0$  is in the positive cone of  $\pi$ .*

*Proof.* We will construct a one-parameter family of operators  $S_R$ , with  $R \geq 0$ , in the positive cone of  $\pi$ , converging to  $T_K$  in the weak operator topology. We start by enumerating the finite set  $A_R = \{g_1, g_2, \dots, g_N\}$ . Now, define  $V_i \subseteq \partial\Gamma^2$  as

$$(5.17) \quad V_i = \Sigma_2(g_i) \setminus \bigcup_{j < i} \Sigma_2(g_j).$$

Since by Proposition 4.5 the double shadows  $\Sigma_2(g_i)$  form a cover, the family  $\{V_i\}_{i \leq N}$  is a measurable partition of  $\partial\Gamma^2$ , and

$$(5.18) \quad \int_{V_i} K d\mu^2 \leq \|K\|_\infty \mu^2(\Sigma_2(g_i)) \prec \omega^{-R}$$

uniformly in  $R$ . If we set

$$(5.19) \quad S_R = \sum_{i=1}^N \int_{V_i} K d\mu^2 \tilde{\pi}(g_i),$$

for any  $\phi, \psi \in \text{Lip}(\partial\Gamma)$  we get, using Lemma 5.3 and the fact that  $V_i \subseteq \Sigma_2(g_i)$

$$(5.20) \quad \begin{aligned} \left| \langle S_R \phi, \psi \rangle - \langle T_K \phi, \psi \rangle \right| &\leq \sum_{i=1}^N \int_{V_i} K d\mu^2 \left| \langle \tilde{\pi}(g_i) \phi, \psi \rangle - \phi(\xi_i) \overline{\psi(\hat{\xi}_i)} \right| + \\ &+ \sum_{i=1}^N \int_{V_i} \left| \phi(\xi_i) \overline{\psi(\hat{\xi}_i)} - \phi(\xi) \overline{\psi(\eta)} \right| K(\xi, \eta) d\mu^2(\xi, \eta) \prec \\ &\prec \|K\|_\infty \frac{\lambda(\phi) \|\psi\|_\infty + \lambda(\psi) \|\phi\|_\infty}{(1 + R)^{1/D}} + \\ &+ \sum_{i=1}^N \int_{V_i} e^{-\epsilon R/2} \left( \lambda(\phi) \|\psi\|_\infty + \lambda(\psi) \|\phi\|_\infty \right) K(\xi, \eta) d\mu^2(\xi, \eta), \end{aligned}$$

so  $\langle S_R \phi, \psi \rangle \xrightarrow{R \rightarrow \infty} \langle T_K \phi, \psi \rangle$ .

Now, observe that for  $\phi \in L^\infty(\partial\Gamma, \mu)$  we have, by Proposition 5.2

$$(5.21) \quad \|S_R \phi\|_\infty \leq \|\phi\|_\infty \sup_{\xi \in \partial\Gamma} \sum_{i=1}^N \int_{V_i} K d\mu^2 \tilde{P}_{g_i}(\xi) \prec \|\phi\|_\infty \omega^{-R} \sup_{\xi \in \partial\Gamma} \sum_{g \in A_R} \tilde{P}_g(\xi) \prec \|\phi\|_\infty$$

uniformly in  $R$  and  $\phi$ . The same argument applies to the adjoint operators  $S_R^*$ , hence the operators  $S_R$  have also uniformly bounded  $L^1$ -norms. By the Riesz-Thorin interpolation theorem they are uniformly bounded on  $L^2(\partial\Gamma, \mu)$ . It follows by density of  $\text{Lip}(\partial\Gamma)$  that  $T_K$  is the weak operator limit of  $S_R$ .  $\square$

## 6. THE BOUNDARY REPRESENTATIONS

In this section we show that the boundary representations are irreducible and weakly contained in the regular representation.

**6.1. Irreducibility.** We will prove irreducibility by using the following classical observation.

**Lemma 6.1.** *Let  $\sigma$  be a unitary representation of a group  $G$  on a Hilbert space  $\mathcal{H}$ . If there exists a cyclic vector  $\phi \in \mathcal{H}$  such that the orthogonal projection  $P_\phi$  onto the subspace  $\mathbb{C}\phi$  is contained in the von Neumann algebra generated by  $\sigma(G) \subseteq \mathcal{B}(\mathcal{H})$ , then the representation  $\sigma$  is irreducible.*

*Proof.* Let  $\mathcal{H}_0 \leq \mathcal{H}$  be a closed nonzero invariant subspace. We may take  $\psi \in \mathcal{H}_0$  with  $\langle \phi, \psi \rangle \neq 0$ , for otherwise

$$(6.1) \quad \langle \sigma(g)\phi, \psi \rangle = \langle \phi, \sigma(g^{-1})\psi \rangle = 0$$

for all  $\psi \in \mathcal{H}_0$  and  $g \in G$ , which by cyclicity of  $\phi$  yields  $\mathcal{H}_0 = 0$ . Since  $P_\phi$  is in the von Neumann algebra generated by  $\sigma(G)$ , the nonzero vector  $P_\phi\psi = \lambda\phi$  belongs to  $\mathcal{H}_0$ . Hence,  $\mathcal{H}_0$  contains a cyclic vector of  $\sigma$  and equals  $\mathcal{H}$ .  $\square$

Proving the irreducibility of the boundary representations now becomes a mere formality.

**Theorem 6.2.** *For any metric  $d \in \mathcal{D}(\Gamma)$  the associated boundary representation is irreducible.*

*Proof.* For a positive function  $\phi \in L^\infty(\partial\Gamma)$  the kernel  $(\xi, \eta) \mapsto \phi(\eta)$  yields a one-dimensional operator  $T$  given by  $T\psi = \langle \psi, \mathbb{1} \rangle \phi$ . By Proposition 5.4 it is contained in the von Neumann algebra of  $\pi$ , and  $T\mathbb{1} = \phi$  is contained in the weakly closed span of  $\pi(\Gamma)\mathbb{1}$ . But  $L^\infty(\partial\Gamma)$  is dense in  $L^2(\partial\Gamma)$ , and weakly closed subspaces are closed, so  $\mathbb{1}$  is a cyclic vector of  $\pi$ . For  $\phi = \mathbb{1}$  the operator  $T$  is the orthogonal projection onto  $\mathbb{C}\mathbb{1}$ , so by Lemma 6.1 we are done.  $\square$

**6.2. Weak containment in the regular representation.** An important property of any unitary representation is whether it is weakly contained in the regular representation. In case of quasi-regular representations we may apply the criterion of [24], which states that if the action of a discrete group  $G$  on a standard Borel space  $(X, \nu)$  is amenable, then all representations obtained by twisting the quasi-regular representation (3.9) of  $G$  on  $L^2(X, \nu)$  with a suitable cocycle are weakly contained in the regular representation.

Amenability of the action on the Gromov boundary was established in [1, Theorem 5.1]. Namely, for any finite Borel measure  $\mu$  on  $\partial\Gamma$ , quasi-invariant under the action of  $\Gamma$ , the action of  $\Gamma$  on  $(\partial\Gamma, \mu)$  is amenable. We thus obtain the following.

**Proposition 6.3.** *The boundary representations of  $\Gamma$  are weakly contained in the regular representation.*

## 7. CLASSIFICATION

Here, we investigate unitary equivalence between representations arising from different metrics on  $\Gamma$ . We show that the corresponding boundary representations are equivalent if and only if the metrics are roughly similar.

For the entire section, let  $d' \in \mathcal{D}(\Gamma)$  be another metric with associated boundary representation  $\pi'$ . Prime will be used to indicate objects associated to  $d'$ , analogous to the ones defined for  $d$ .

**7.1. Preparatory lemmas.** We begin by extending the scope of Proposition 5.4 to orthogonal projections onto subspaces  $L^2(E, \mu)$ . This will be used to show that any unitary isomorphism of boundary representations preserves this family of projections and thus has to be induced by a map of boundaries.

**Lemma 7.1.** *For any measurable set  $E \subseteq \partial\Gamma$  the orthogonal projection  $P_E$  onto  $L^2(E, \mu) \subseteq L^2(\partial\Gamma, \mu)$  is contained in the positive cone of  $\pi$ .*

*Proof.* Fix  $E \subseteq \partial\Gamma$ . For  $\theta > 0$  define  $K_\theta \in L^\infty(\partial\Gamma^2, \mu^2)$  as

$$(7.1) \quad K_\theta(\xi, \eta) = \frac{1}{\mu(B_{\partial\Gamma}(\xi, \theta))} \chi_E(\xi) \chi_{B_{\partial\Gamma}(\xi, \theta)}(\eta)$$

and let  $T_\theta$  be the operator with kernel  $K_\theta$ . By Proposition 5.4 it is contained in the weak operator closure of the cone spanned by  $\pi(\Gamma)$ . For  $\phi, \psi \in \text{Lip}(\partial\Gamma)$  we have

$$(7.2) \quad \begin{aligned} & \left| \langle T_\theta \phi, \psi \rangle - \langle P_E \phi, \psi \rangle \right| \leq \\ & \leq \left| \int_E \phi(\xi) \frac{1}{\mu(B_{\partial\Gamma}(\xi, \theta))} \int_{B_{\partial\Gamma}(\xi, \theta)} (\overline{\psi(\eta)} - \overline{\psi(\xi)}) d\mu(\eta) d\mu(\xi) \right| \leq \\ & \leq \mu(E) \|\phi\|_\infty \lambda(\psi) \theta \xrightarrow{\theta \rightarrow 0} 0, \end{aligned}$$

so to finish the proof it only remains to show that  $T_\theta$  are uniformly bounded. We may estimate their operator norms using the Schwarz inequality, obtaining

$$(7.3) \quad \|T_\theta\| \leq \left\| \int_{\partial\Gamma} K_\theta(\xi, \eta) d\mu(\xi) \right\|_\infty \left\| \int_{\partial\Gamma} K_\theta(\xi, \eta) d\mu(\eta) \right\|_\infty.$$

By Ahlfors regularity we get

$$(7.4) \quad \int_{\partial\Gamma} K_\theta(\xi, \eta) d\mu(\xi) = \int_E \frac{\chi_{B_{\partial\Gamma}(\xi, \theta)}(\eta)}{\mu(B_{\partial\Gamma}(\xi, \theta))} d\mu(\xi) \asymp \int_E \frac{\chi_{B_{\partial\Gamma}(\eta, \theta)}(\xi)}{\mu(B_{\partial\Gamma}(\eta, \theta))} d\mu(\xi) \leq 1,$$

while the second integral is simply equal to  $\chi_E(\xi) \leq 1$ .  $\square$

A measure  $\nu$  on  $X$  will be called *almost invariant* under the action of a group  $G$  if it is quasi-invariant and the estimate

$$(7.5) \quad \frac{dg_*\nu}{d\nu}(x) \asymp 1$$

holds uniformly in  $g \in G$  and  $x \in X$ .

**Lemma 7.2.** *The measure  $\nu = d_\epsilon(\xi, \eta)^{-2D} d\mu^2(\xi, \eta)$  is an almost invariant measure on  $\partial\Gamma^2$ .*

*Proof.* The logarithm of the Radon-Nikodym derivative of the pushforward of  $\nu$  satisfies a.e.

$$(7.6) \quad \log_\omega \left( \frac{dg_*\nu}{d\nu}(\xi, \eta) \right) = \log_\omega \left( \left( \frac{d_\epsilon(g^{-1}\xi, g^{-1}\eta)}{d_\epsilon(\xi, \eta)} \right)^{-2D} P_g(\xi) P_g(\eta) \right) \approx \\ \approx 2(g^{-1}\xi, g^{-1}\eta) - 2(\xi, \eta) + 2(g, \xi) + 2(g, \eta) - 2|g| \approx 0,$$

where the last estimate is obtained by expanding the definition of the Gromov product, after replacing  $\xi$  and  $\eta$  by sequences  $x_n \rightarrow \xi$  and  $y_n \rightarrow \eta$ .  $\square$

**7.2. Equivalence in terms of measurable structures.** By an *isomorphism* of measure spaces  $(X, \mu_X)$  and  $(Y, \mu_Y)$  we will understand a Borel map  $F: X \rightarrow Y$ , for which there exist two subsets  $N \subseteq X$  and  $N' \subseteq Y$  of measure 0, such that the restriction  $F: X \setminus N \rightarrow Y \setminus N'$  is a Borel isomorphism, and the pushforward  $F_*\mu_X$  is equivalent to  $\mu_Y$ . Such an isomorphism will be called *equivariant* if the corresponding equivariance condition is satisfied almost everywhere.

**Lemma 7.3.** *Suppose that the representations  $\pi$  and  $\pi'$  are unitarily equivalent. Then there exists a  $\Gamma$ -equivariant isomorphism  $F: (\partial\Gamma, \mu) \rightarrow (\partial\Gamma, \mu')$ .*

*Proof.* Suppose  $T: L^2(\partial\Gamma, \mu) \rightarrow L^2(\partial\Gamma, \mu')$  is a unitary intertwining operator. It induces a  $\Gamma$ -equivariant isomorphism of von Neumann algebras

$$(7.7) \quad \hat{T}: \mathcal{B}(L^2(\partial\Gamma, \mu')) \rightarrow \mathcal{B}(L^2(\partial\Gamma, \mu)),$$

endowed with the conjugation actions by the corresponding representations, given by  $\hat{T}(S) = T^*ST$ , which takes the positive cone of  $\pi'$  onto the positive cone of  $\pi$ , and preserves orthogonal projections.

Now, observe that the subalgebra  $L^\infty(\partial\Gamma, \mu') \leq \mathcal{B}(L^2(\partial\Gamma, \mu'))$  of multiplication operators is generated by the orthogonal projections  $P_E$  onto  $L^2(E, \mu')$ , which can be characterized as the orthogonal projections  $P$  such that both  $P$  and  $I - P$  are in the positive cone of  $\pi'$ . One inclusion is a consequence Lemma 7.1. For the other one observe that if both  $P$  and  $I - P$  are in the cone, then they preserve the natural partial order on functions in  $L^2(\partial\Gamma, \mu_\delta)$ . Since the only possible decompositions  $\mathbb{1} = P\mathbb{1} + (I - P)\mathbb{1}$  into a sum of two orthogonal positive functions are of the form  $\mathbb{1} = \chi_E + \chi_{E^c}$ , for bounded positive  $\phi$  we get

$$(7.8) \quad P\phi \leq P(\|\phi\|_\infty \mathbb{1}) \leq \|\phi\|_\infty \chi_E,$$

and similarly for  $I - P$ , which implies that  $P$  is the projection  $P_E$ . It follows that  $\hat{T}$  restricts to a  $\Gamma$ -equivariant isomorphism between  $L^\infty(\partial\Gamma, \mu')$  and  $L^\infty(\partial\Gamma, \mu)$ . By spectral theory, such an isomorphism is induced by a  $\Gamma$ -equivariant isomorphism  $F: (\partial\Gamma, \mu) \rightarrow (\partial\Gamma, \mu')$ .  $\square$

**7.3. Equivalence in terms of metric structures.** Recall that in a metric space  $(X, d)$  the cross-ratio of a quadruple of distinct points  $x, y, z, w \in X$  is defined as

$$(7.9) \quad [x, y, z, w] = \frac{d(x, z)d(y, w)}{d(x, w)d(y, z)}.$$

The next lemma is an adaptation of a classical argument from ergodic theory (see e.g. the proof of [18, Theorem 6.2]); for the sake of self-containment we include the detailed proof.



**Lemma 7.4.** *If  $\pi$  and  $\pi'$  are unitarily equivalent, then we have*

$$(7.10) \quad d'_\epsilon(\xi, \eta) \asymp d_\epsilon(\xi, \eta)^{D/D'}.$$

*Proof.* By Lemma 7.3, there exists a  $\Gamma$ -equivariant isomorphism  $F: (\partial\Gamma, \mu) \rightarrow (\partial\Gamma, \mu')$ . Let  $\nu$  and  $\nu'$  be the almost invariant measures on  $\partial\Gamma^2$ , corresponding to the metrics  $d$  and  $d'$ . Then the pushforward  $F_*^2\nu$  is another almost invariant measure on  $\partial\Gamma^2$ , equivalent to  $\nu'$ . We have

$$(7.11) \quad \frac{dF_*^2\nu}{d\nu'} \circ g^{-1} = \frac{d(gF^2)_*\nu}{dF_*^2\nu} \cdot \frac{dF_*^2\nu}{d\nu'} \cdot \frac{d\nu'}{d g_*\nu'} \asymp \frac{dF_*^2\nu}{d\nu'}$$

for all  $g \in \Gamma$ , and therefore

$$(7.12) \quad \sup_{g \in \Gamma} \frac{dF_*^2\nu}{d\nu'} \circ g^{-1} \prec \frac{dF_*^2\nu}{d\nu'} \prec \inf_{g \in \Gamma} \frac{dF_*^2\nu}{d\nu'} \circ g^{-1}.$$

Both bounds are  $\Gamma$ -invariant functions, hence by double ergodicity of the Patterson-Sullivan measures, they are constant a.e., so finally, by expanding the Radon-Nikodym derivative  $dF_*^2\nu/d\nu'$ , we get

$$(7.13) \quad d'_\epsilon(\xi, \eta)^{-2D'} \asymp d_\epsilon(F^{-1}(\xi), F^{-1}(\eta))^{-2D} \frac{dF_*\mu}{d\mu'}(\xi) \frac{dF_*\mu}{d\mu'}(\eta)$$

a.e. After raising this to the power  $-1/2D'$  and plugging into the definition of the cross-ratio, the Radon-Nikodym derivatives cancel out, and we obtain that the estimate

$$(7.14) \quad [F(\xi_1), F(\xi_2), F(\eta_1), F(\eta_2)] \asymp [\xi_1, \xi_2, \eta_1, \eta_2]^{D/D'}$$

holds on a subset  $E \subseteq \partial\Gamma^4$  of measure 1 (with respect to  $\mu^4$ ).

If we take  $\theta > 0$  and  $(\xi_2, \eta_2)$  such that the corresponding section  $\{(\xi_1, \eta_1) : (\xi_1, \xi_2, \eta_1, \eta_2) \in E\}$  has measure 1, we get

$$(7.15) \quad d'_\epsilon(F(\xi_1), F(\eta_1)) \prec_{\xi_2, \eta_2, \theta} d_\epsilon(\xi_1, \eta_1)^{D/D'}$$

for  $(\xi_1, \eta_1)$  in a subset of full measure in  $B_{\partial\Gamma}(\eta_2, \theta)^c \times B_{\partial\Gamma}(\xi_2, \theta)^c$ . Finitely many such sets cover  $\partial\Gamma^2$ , and by Fubini theorem,  $(\xi_2, \eta_2)$  can be chosen from a set of measure 1, so the estimate (7.15) actually holds uniformly on a subset  $E' \subseteq \partial\Gamma^2$  of measure 1. Now, denote by  $E''$  the set consisting of  $\xi \in \partial\Gamma$  such that the section  $E'_\xi = \{\eta : (\xi, \eta) \in E'\}$  has measure 1. For  $\xi, \eta \in E''$  and  $\zeta \in E'_\xi \cap E'_\eta$  we have

$$(7.16) \quad \begin{aligned} d'_\epsilon(F(\xi), F(\eta)) &\leq d'_\epsilon(F(\xi), F(\zeta)) + d'_\epsilon(F(\eta), F(\zeta)) \\ &\prec d_\epsilon(\xi, \zeta)^{D/D'} + d_\epsilon(\eta, \zeta)^{D/D'}. \end{aligned}$$

If we let  $\zeta$  converge to  $\eta$  from within the dense set  $E'_\xi \cap E'_\eta$ , we get a Hölder estimate

$$(7.17) \quad d'_\epsilon(F(\xi), F(\eta)) \prec d_\epsilon(\xi, \eta)^{D/D'}$$

on  $F$ , satisfied a.e. on  $\partial\Gamma$ . This implies that  $F$  is equal a.e. to a continuous map  $H: \partial\Gamma \rightarrow \partial\Gamma$ . By symmetry, from  $F^{-1}$  we may construct a continuous inverse of  $H$ , so it is a homeomorphism. Equivariance is clear, and by the remark in Section 2.4,  $H$  is in fact the identity map. Estimate (7.10) follows from (7.17) and symmetry.  $\square$

Now we are ready to state and prove the equivalence result.

**Theorem 7.5.** *Let  $d, d' \in \mathcal{D}(\Gamma)$  give rise to boundary representations  $\pi$  and  $\pi'$ . Then  $\pi$  and  $\pi'$  are unitarily equivalent if and only if  $d$  and  $d'$  are roughly similar.*

*Proof.* First, observe that for the metrics  $d$  and  $d'$ , being roughly similar means exactly that  $d \approx Ad'$ . If this is satisfied, the visual metrics  $d_\epsilon$  and  $d'_{A\epsilon}$  are bi-Lipschitz equivalent. Hence, the corresponding Hausdorff measures are equivalent, and the boundary representations are equivalent by the discussion in Section 3.3.

For the other implication, we use Lemma 7.4 to get the estimate (7.10) on the visual metrics, which implies that the Hausdorff measures  $\mu$  and  $\mu'$  are equivalent with Radon-Nikodym derivatives bounded away from 0 and  $\infty$ . Hence,  $\mu$  is  $\Gamma$ -quasi-conformal with respect to the metric  $d'$ , and in consequence

$$(7.18) \quad \omega^{2(g, \xi) - |g|} \asymp \omega'^{2(g, \xi)' - |g|'}$$

uniformly in  $g$  and  $\xi$ . By taking the logarithms of both sides, and then suprema over  $\xi$ , using Lemma 4.1 we obtain

$$(7.19) \quad |g| \approx \frac{\log \omega'}{\log \omega} |g|' = \frac{D'}{D} |g|',$$

which ends the proof.  $\square$

*Remark 7.6.* It might be tempting to try to extend the class of boundary representations even further, by allowing  $d$  to be a pseudometric. For instance, if  $\Gamma$  acts properly and cocompactly on a space  $X$  then the orbit map in general induces a pseudometric on  $\Gamma$ . However, pseudometrics do not lead to any new representations. In fact, any such pseudometric  $d$  is roughly isometric to a metric

$$(7.20) \quad d^+(g, h) = \begin{cases} d(g, h) + 1 & \text{for } g \neq h \\ 0 & \text{for } g = h. \end{cases}$$

## 8. EXAMPLES

Here we apply the obtained results to some classes of groups appearing in nature. It is less self-contained than the preceding sections—for more details the reader is referred to the appropriate literature.

**8.1. Fundamental groups of negatively curved manifolds.** The following setting was studied by Bader and Muchnik in [3]. Let  $M$  be a closed Riemannian manifold with strictly negative curvature. Its universal cover  $\tilde{M}$  is then a hyperbolic metric space, on which  $\Gamma = \pi_1(M)$  acts freely and cocompactly by isometries. This action thus extends to an action of  $\Gamma$  on the Gromov boundary  $\partial\tilde{M}$ , and yields a unitary representation defined by formula (3.9). Any orbit map  $\Gamma \rightarrow \tilde{M}$  induces a hyperbolic metric  $d \in \mathcal{D}(\Gamma)$ , and since we may identify  $\partial\Gamma$  with  $\partial\tilde{M}$ , the resulting representation is actually the boundary representation of  $\Gamma$  associated to the metric  $d$ .

The group  $\Gamma$  may of course appear as the fundamental group of many non-isometric Riemannian manifolds, and any such realization leads to a potentially different representation. The main theorems of [3] state that all these representations are irreducible, and that they are equivalent if and only if the marked length spectra of the corresponding manifolds are proportional. By the marked length spectrum of  $M$  we understand the function  $\ell: \pi_1(M) \rightarrow (0, \infty)$ , which to every  $[\gamma] \in \pi_1(M)$  assigns the length of the unique geodesic loop freely homotopic to  $\gamma$ .

The irreducibility of the Bader-Muchnik representations is a special case of Theorem 6.2. To conclude the equivalence condition of [3] using Theorem 7.5, it is enough to observe that proportionality of the marked length spectra is equivalent to rough similarity of the induced metrics on  $\pi_1(M)$ . In one direction it is trivial—the length of the shortest geodesic loop in the free homotopy class of  $g \in \pi_1(M)$  is the translation length of  $g$  acting on  $\tilde{M}$ , and can be expressed in terms of the metric as

$$(8.1) \quad \ell(g) = \lim_{n \rightarrow \infty} \frac{|g^n|}{n}.$$

For the other direction we may resort to [26, Theorem 2.2], which states that the marked length spectrum determines the cross-ratio on the boundary. Hence, proportional marked length spectra lead to cross-ratios satisfying  $[\cdot]' = [\cdot]^\alpha$ , and thus they arise from roughly similar metrics.

**8.2. Green metrics and Poisson boundaries.** Let  $\Gamma$  be a non-elementary hyperbolic group, and let  $\nu$  be a symmetric probability measure on  $\Gamma$ , whose support generates  $\Gamma$ . Such a measure gives rise to a random walk on  $\Gamma$ ; denote by  $F(g, h)$  the probability that starting at  $g$ , it ever reaches  $h$ . Assume that  $\nu$  has *exponential moment*, i.e. there exists  $\lambda > 0$  such that

$$(8.2) \quad \sum_{g \in \Gamma} e^{\lambda|g|} \nu(g) < \infty,$$

where the length is taken with respect to any word metric on  $\Gamma$ , and that for any  $r$  there exists a constant  $C(r)$  for which

$$(8.3) \quad F(x, y) \leq C(r)F(x, v)F(v, y)$$

for  $v$  within distance  $r$  from a geodesic (again, with respect to some fixed word metric) joining  $x$  and  $y$ . These two conditions hold for any finitely supported measure [8, Corollary 1.2], and ensure that the *Green metric*

$$(8.4) \quad d_G(g, h) = -\log F(g, h),$$

which was introduced in [6], and studied further in [7], belongs to the class  $\mathcal{D}(\Gamma)$ .

Trajectories of the random walk with law  $\nu$  almost surely converge to a point in  $\partial\Gamma$ , and the hitting probability defines the *harmonic measure*  $\hat{\nu}$  on  $\partial\Gamma$  associated with  $\nu$ . By [8, Theorems 1.1(ii) and 1.5], it turns out that  $\hat{\nu}$  is equivalent to the Patterson-Sullivan measure associated with the Green metric  $d_G$ , and thus yields the same quasi-regular representation. On the other hand, a measure with exponential moment has finite first moment, and by [22, Theorem 7.4], the Gromov boundary with the harmonic measure  $(\partial\Gamma, \hat{\nu})$  is actually isomorphic to the Poisson boundary of  $(\Gamma, \nu)$ . By Theorem 6.2, we therefore obtain

**Theorem 8.1.** *Let  $\nu$  be a symmetric measure on  $\Gamma$ , satisfying conditions (8.2) and (8.3). Then the quasi-regular representation associated with the Poisson boundary  $(\partial\Gamma, \hat{\nu})$  is irreducible.*

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